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Investigating Representations of Ratio among Prospective Mathematics Teachers: a Study of Student-Teachers and Students of Mathematics in an Irish University

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Abstract

This paper contributes to the ATEE Ratio Project, which aims to consider prospective teachers' understanding of ratio through the meanings and representations they associate with it. The paper describes a study undertaken in Ireland. Some modifications were made to the original instrument with a view to eliciting a wider range of meanings and representations than those obtained from Irish participants in the initial phase of the study. Responses from three small groups of prospective teachers – the original Irish participants and two other groups – are compared. Recommendations are made with regard to further development of the instrument and its use in teacher education as well as for research.

Keywords: Representation, Ratio, Teacher Knowledge

Introduction

The concept of ratio and the allied area of proportional reasoning are important throughout school mathematics. However, they are problematic topics, with research providing evidence that they cause difficulties for many students and also for some teachers. This latter aspect is a cause for particular concern, as it is likely to perpetuate the problems encountered at student level. Research is needed in order to investigate teacher knowledge further and to find ways of enhancing that knowledge through appropriate teacher education.

At the 2011 annual conference of ATEE, the Science and Mathematics Education Research and Development Community initiated a multiple case-study of prospective teachers' content knowledge of ratio for teaching mathematics and science. An instrument was developed in which students were asked about the meanings they ascribed to ratio and the representations they associated with it. Data were collected at four institutions, including one in Ireland. Analysis of the meanings and representations led to the conjecture that participants who offered meanings reflecting two variables, and who provided many, varied, and relevant representations, possessed *relational understanding* of ratio. The initial findings of the 'Ratio Project' were reported at the 2012 conference.

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The present paper contributes to the study by undertaking a further round of data collection in 2013 in the Irish institution. The participants were student-teachers taking a Mathematics Pedagogy module in their teacher education programme – the equivalent group to that in the original study – and mathematics undergraduates taking a module on Mathematics Education. Collectively, the participants can be labelled for the purposes of the study as 'prospective teachers.' Initial research questions included:

- A: What are the prospective Irish teachers' representations of ratio, and how do these differ among the three groups (the 2012 group and the two 2013 groups)?
- B: Do the data provide support for the Ratio Project conjecture on relational understanding?

Additionally, the paper focuses on the performance of the instrument used for data collection, with a view to undertaking further work in the area.

In the next two sections of the paper, relevant literature is reviewed and the ATEE Ratio Project is described. The context for Irish work on the project is provided in the following section. The remaining three sections deal respectively with methodology, findings and discussion, and conclusions.

Literature review

Four relevant fields of literature are discussed here: understanding in mathematics education; the role of representations; problems with ratio and proportional thinking; and variation in definitions of ratio and proportion. In the case of the first three, key research from the 1970s or 1980s is identified, and an outline is provided of subsequent developments in the field. The fourth topic is included to document some need for clarification of terms, particularly in an international study.

Understanding in mathematics education

A seminal paper by Richard Skemp (1976) distinguished two forms of understanding, 'relational' and 'instrumental.' Relational understanding is understanding 'why' and being aware of the connections between concepts in the conceptual structures that permeate mathematics; instrumental understanding is understanding 'how', hence being able to carry out procedures or demonstrate skills. Skemp's work was referenced in the influential Cockcroft Report, produced by a committee of inquiry into teaching mathematics in England and Wales (Cockcroft 1982). Of relevance here is the report's identification of three elements in the teaching of mathematics: facts and skills, conceptual structures, and general strategies – needing to be taught (and learnt) in distinctively different ways. The report endorses the achievement of 'fluency' in skills, but is opposed to 'rote' memorisation of anything that can be learned in a meaningful way. It emphasises the developmental nature of understanding and points, albeit somewhat indirectly, to the mutual development of Skemp's two forms (Cockcroft 1982, 69-71). In the USA, Hiebert's (1986) work on conceptual and procedural knowledge reflects Skemp's on relational and instrumental understanding, although using different terminology.

The debate between prioritising skills and prioritising (relational) understanding, discussed in the Cockcroft report, has continued – notably in the 'Math(s) Wars' that have flourished when and where curricula have been reformed (Abbott et al. 2010). In a major summary of research, Hiebert and Carpenter (1992) recognised the interdependence of skills and understanding but tended to point to development of understanding as the leading partner; other publications emphasise terms such as 'sense making' and 'meaning making' that reflect relational understanding (see for example Hiebert et al. 1997). However, by the turn of the century, a stronger focus on the joint value of the two forms of understanding was coming to prominence. The National Research Council (NRC) study *Adding it up: Helping children learn mathematics* specified five 'strands' of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition, with the choice of the word 'strand' highlighting their interdependence (NRC 2001).

These trends are mirrored in curriculum specifications. The National Council of Teachers of Mathematics (NCTM) document *Principles and Standards for School Mathematics* lists ten standards describing 'a *connected* body of mathematical understandings and competencies' indicating what students should know and be able to use (NCTM 2000, emphasis added). The more recent Common Core State Standards in the USA 'define what students should understand and be able to do in their study of mathematics…. Mathematical understanding and procedural skill are equally important' (Common Core State Standards Initiative 2010, 4). In Ireland, the latest versions of objectives for the – currently evolving – second level mathematics curriculum, 'Project Maths' (for grades 7-12), use the NRC (2001) definition of the development of proficiency; this has replaced Skemp's (1976) terminology which had been employed in all previous second level syllabuses since 1990. It is perhaps important to indicate that older syllabuses also referred to understanding and skill, but the specifications and the relationship between them were based on – dare one say! – common sense and experience rather than being grounded in formal research (see for example Department of

Education 1987, 54; Department of Education and Science 2000, 3; Department of Education and Skills 2013, 6).

Work on understanding usually refers tacitly or explicitly to the relational form, and much attention has been focused on how students can develop it. One way, of particular relevance to this paper, is by use of *representations*. Their role is now discussed.

The role of representations

Important work on representations was carried out from the early 1980s, and key aspects were discussed at and after a symposium held in 1984 (Janvier 1987). Lesh, Post and Behr (1987) characterised five different types of representation: experience-based 'scripts' in which knowledge is organized round real-world events; manipulatable models such as Cuisenaire rods encapsulating the mathematical relationships between elements; pictures and diagrams; spoken languages, including specialized sublanguages such as that for logic; and written symbols, again involving specialized symbol systems as well as ordinary language.

Subsequent research from the late 1980s and 1990s is drawn together in a paper by Pape and Tscoshanov (2001) entitled 'The role of representation(s) in developing mathematical understanding.' The 's' in parentheses highlights that 'representation' can be a process or a product. They state: 'It is now well accepted that the use of particular modes of representations (e.g. visual or concrete) leads to improvement of students' mathematical abilities and development of their advanced problem solving and reasoning skills… That is, the use of multiple representations facilitates students' development of mathematical concepts' (Pape and Tscoshanov 2001, 120). They also emphasise the value of discourse among peers and teachers to negotiate and refine understanding (p. 124). Crowley and Tall (2006) emphasise the value of understanding links to and between procedures, as well as links between graphical and symbolic representations, again underlining the interdependence of relational and instrumental understanding.

Growing recognition of the importance of representation in curriculum specification, as well as in research, is indicated by the fact that representation is one of the five process standards in the NCTM's *Principles and Standards.* It states that all students should be enabled to create representations, use them to communicate mathematical ideas, and translate among representations to solve problems. The fundamental importance of representations for how people understand and use mathematical ideas is emphasised

(NCTM 2000). In Ireland, the language of representations has been introduced in the Project Maths curriculum, a development highlighted below.

Problems with ratio and proportional thinking

Work on ratio from the 1970s and 1980s, notably by Hart and her associates in the UK, identified problems in responses from both primary and secondary students (Hart et al. 1989). The study of ratio and of representations was brought together explicitly by Lesh, Behr and Post (1987) in work that highlighted students' problems in moving between representations. Subsequently, major syntheses of research on ratio-related issues were produced by Behr et al. (1992) and Lamon (2007). Lamon's account emphasises changes in the field since 1992, and ends with a long list of unanswered but researchable questions. Livy and Vale (2011) provide a more recent summary of evidence that students in the middle years of schooling have poor understanding of ratio and proportional reasoning.

With regard to teacher knowledge, Lamon (2007) refers briefly to research in the area. Later work, particularly in Australia, is relevant here. In their study of 297 prospective teachers, Livy and Vale (2011) found low levels of correct responses to relevant ratio and proportion test items. Chick's (2010) study of 40 practising secondary teachers identified some deficiencies in their knowledge for teaching ratio. The continuing importance of addressing knowledge for teaching the topic is indicated by the inclusion of relevant articles in the NCTM teacher journals (for instance Jarvis 2007; Rutchie and Bennett 2013).

Variation in definitions of ratio and proportion

A challenge in doing research on ratio and proportion lies in the fact that the underlying concepts are not always clearly defined (Lamon 2007). While it may be unproblematic to state that a ratio is a comparison of (or relationship between) two numbers measured in the same units (for example Jarvis 2007), there is disagreement on the extent to which part-whole relationships should be accepted as ratios. Following work by Clark, Berenson, and Cavey (2003), preference may be given to models that clearly involve two distinct variables, while part-whole relationships may be situated in the intersection between ratio and fraction.

As regards proportion, there appear to be cultural differences in the definitions. American literature typically refers to 'a proportion' as an equality between two ratios (see for example Jarvis 2007). In Irish and also English curricula, that formal definition is not used; the typical emphasis is on, say, the proportion of orange in an orange-and-water drink being such-and-such, expressed as a fraction or percentage (Suggate, Davis and Goulding 2006, 86).

The ATEE Ratio Project

As pointed out in the Introduction, the Ratio Project was initiated at the 2011 ATEE conference, and initial findings were reported at the 2012 conference (Berenson et al. 2013). As the present paper builds heavily on those findings, the work is described in some detail.

For the initial phase of the study, the following research questions were chosen:

(a) What meanings do prospective teachers at primary and secondary levels in [specific institutions in the participants' home countries] give to the term 'ratio'?

(b) What multiple representations do these prospective teachers associate with the term 'ratio'?

(c) Do the prospective teachers' descriptive meanings and representations indicate different levels of understanding for teaching ratio?

Participating members of the RDC devised a one-page instrument that could be administered within ten to fifteen minutes in appropriate classes: typically, of students in teacher education courses. Five items were presented:

1. What does the term 'ratio' mean to you?

2a. When do you use ratios?

2b. Who else uses ratios?

3. How do you represent a ratio using mathematical symbols?

4. Draw several representations of how ratios are used.

Localised versions of the instrument, reflecting differences in language and in the structure of the education systems, were prepared to allow for data collection in three countries.

Data from 158 students (including 16 Irish students) were collected and analysed. Initial examination showed that some participants provided rich explanations and illustrations, but the responses of some others were brief and relatively thin. Also, participants in general did not relate ratio to more advanced topics in the curriculum, such as trigonometry or rate of change; many instances, including ones offered by prospective secondary as well as prospective primary teachers, referred to concepts usually addressed in middle school grades.

Further analysis, undertaken using a grounded theory approach, focused chiefly on responses to items 1 and 4. From the meanings specified and the representations provided, three emergent themes with regard to *meanings* were identified (research question (a)). Some descriptions or representations emphasised or allowed the inference that the participants' concepts includes the notion of two distinct variables; some appeared to refer to uses or applications or special types of ratio; and some related to part-whole relationships. The themes can be labelled for convenience as 'two variables,' 'applications' and 'part-whole'; examples of key aspects to which participants referred are given in Table 1. In line with their preferred definition of ratio as involving two distinct variables (see above), the research team regarded responses classified as belonging to the third theme – ratio as meaning part-whole relationships – as tending to point to a lower level of understanding than those from the two-variable theme, especially if the part-whole meaning was presented alone. [Table 1 near here].

Table 1: Emergent themes for participants' descriptions of the meanings they ascribed to ratio

Two variables	**Applications**	**Part / whole**
Comparison	Rate	Fraction
Relationship	Scale	Decimal
	Odds	Percent?
	Proportion	
	Division / splitting	
	Percent	

With regard to *representations* (research question (b)), the responses to item 4 displayed considerable variety. As well as typically reflecting one or more of the three themes, participants used different kinds of representation; some provided 'drawings' or other pictorial representations, as suggested by the formulation of the item, while others offered only symbols or words. Moreover, some participants made no response at all to this item. The word 'draw' may have suggested that verbal or symbolic responses were unacceptable, whereas a definition of representations as *any ideas associated with another idea in mathematics that is written, drawn, or spoken* includes these representations (see the classification by Lesh, Post and Behr (1987) cited above).

It remained to consider research question (c). Skemp's (1976) work on relational and instrumental understanding offered a way of categorising different levels of understanding, with a connection to multiple representations being provided by other research cited above. The research team conjectured that participants using more representations, and especially representations of different types, were displaying more relational understanding (Table 2). However, the team members were not able to make conjectures around participants' instrumental understanding, as none of the items in the instrument asked them to carry out a procedure. [Table 2 near here]

Table 2: Conjectured indicators of presence or absence of relational understanding

Displays relational understanding	**Does not display relational understanding**
Meaning of ratio reflects two variables	Meaning of ratio does not reflect two variables
Provides many representations	Provides few representations
Uses multiple types of representation	Uses few types of representation
Cites / draws relevant applications	Provides symbolic representations only

The responses to item 3 did not contribute significantly to the investigation of understanding. The question as formulated asked participants to display factual knowledge, rather than understanding of why or how; it could be answered satisfactorily by provision of one (correct) representation, such as the colon symbol. Nonetheless, useful information with regard to understanding could sometimes be gleaned from more extended responses or from incorrect ones. Responses to item 2 were not considered in the paper.

Initial findings in grounded theory need to be tested with different samples to see if they are confirmed or contradicted. Work on the Ratio Project has therefore continued in the academic year 2012-2013.

Context of the Irish study

The follow-up study reported in this paper is part of the continuing work on the project. The authors were especially interested in carrying out further rounds of data collection in Ireland because of indications that ratio as a topic and proportional reasoning as a 'golden thread' have been given rather less attention in Irish curricula than is the case in some other countries (Oldham 2013). Ratio is mentioned only very briefly in the curriculum for the final grade in the primary curriculum (grades preK-6), while 'ratio and proportion' has typically been a one-line entry in successive versions of second level curricula (grades 7-12). The latest curriculum change at second level – Project Maths – has improved the situation somewhat, at least as regards the intended curriculum, but the effects (if any) have yet to impact on university courses. Additionally, in the experience of the first author – extending over several decades – ratio and proportional reasoning have not been problematised in Irish discussions on mathematics education; moreover, according to the second author, who is involved in the professional development courses accompanying the rollout of the new curriculum, there has still been little or no explicit focus on ratio and proportional reasoning in its implementation. In both authors' experience, the concept, or at least the language, of representations and the provision of tasks involving students *creating* their own representations (compare NCTM 2000) have also been accorded little emphasis over the years, though – as mentioned above – the Project Maths curriculum does make some use of the language of representations (see for example DES 2013). Participation by mathematics education lecturers and students in a larger study could not only provide more insight into Irish student-teachers' understanding of ratio, but could also help to open up a general Irish discussion on ratio and proportional thinking.

The lack of focus on ratio and related concepts, together with unfamiliarity with the language of representations, may explain why some of the responses from the Irish cohort in 2012 were among those that were thin rather than rich. Further investigation with different cohorts is therefore relevant, with the twofold aim of *preparing for a larger Irish study* and *testing the operation and findings of the ATEE Ratio study with different samples and/or populations of participants*.

Methodology

This section addresses three issues. These are: changes to the instrument; choice of the sample and refinement of the research questions; and data collection and analysis.

Changes to the instrument

In view of limitations of the instrument emerging from the initial phase of the study, as described above, it was decided to amend the existing items 3 and 4 in the hope of attracting a wider range of responses. Thus, the items were reformulated as follows:

- Item 3: 'How do you represent a ratio using mathematical symbols? If relevant, indicate clearly which is/are the **main** symbol(s) but list others as well. *You may write expressions that include the symbols, rather than just the symbols themselves.*'
- Item 4: 'What representations – drawings, charts, graphs, words, and so forth – might you use to explain ratio and show how it is used? *Present your ideas here and/or overleaf as you wish.*'

Extensions to the scope of the instrument to capture instrumental understanding were considered. An item on instrumental understanding might contain an example or examples for the respondents to address, or else ask respondents to provide a 'ratio calculation' of their choice. However, the former approach might skew the responses to other items towards the symbols or representations used; the latter, while perhaps encouraging creativity and giving further insight into relational understanding, could add significantly to the time taken for administering the instrument. In either case, the additional item could detract from the usability of the instrument with future cohorts in time-pressurised education programmes. It was therefore decided to make no further changes at this stage.

Choice of the sample and refinement of the research questions

Two groups, both from the institution at which the authors teach, were chosen for participation in the 2013 study. They were graduates taking Mathematics Pedagogy in their teacher education course, the Professional Diploma in Education (PDE) – the equivalent group to the Irish participants in the original study – and Mathematics undergraduates taking a module on Mathematics Education as part of the third or fourth year of their degree programme. The Mathematics Education module requires students to help in school classrooms, entitling them to be called 'prospective teachers' in this study. Typically, Mathematics Pedagogy groups contain some students who have done only a little mathematics in their degree programmes (for instance, in Biology or Business), whereas the Mathematics students are specialising in the subject or taking it as a major component of their studies. Some difference in the overall levels of understanding of the two groups can be hypothesised.

As analysis of the Irish data from the original study had been only briefly reported separately from the other international data (Oldham 2013), the present study focuses on data from all three groups: henceforth referred to as PDE 2012, PDE 2013 and Maths 2013. Research question A, as set out in the introduction, was therefore refined as follows.

(Ai) Is the range of representations obtained using the amended instrument greater than before?

If so, in what ways?

(Aii) In other respects, are the PDE 2013 students' responses similar to those of the PDE 2012 group?

(Aiii) What are the similarities and differences between the responses of the PDE 2013 and Maths 2013 groups?

Data collection and analysis

The instrument was given to the Mathematics Pedagogy class at the start of a lecture (as had been done the previous year), and to the Mathematics education class at the end of a lecture. A period of ten to fifteen minutes was scheduled for the exercise in each case. It can be noted here that the PDE 2013 students finished comfortably in the allotted time – as their predecessors had done in 2012 – but that several of the Maths 2013 students did not; they completed the work outside class time.

The data collected in 2013 were summarised and coded using a procedure similar to that in 2012 (Berenson et al. 2013). Thus, for item 1, occurrences of each entry in Table 1 above were counted for each of the 2013 participants; similar tallies were made for items 3 and 4. Item 2 was not considered for this paper. The 2012 and 2013 data were then analysed together. The small size of the groups, and also the differences in the instrument (between the 2012 and 2013 groups) and administration time (between the Maths and PDE groups), limited the types of analysis undertaken, as described below.

Findings

A total of 33 completed instruments was received in 2013. The distribution of participants over the two years is shown in Table 3. [Table 3 near here]

Table 3: Numbers of participants, by group

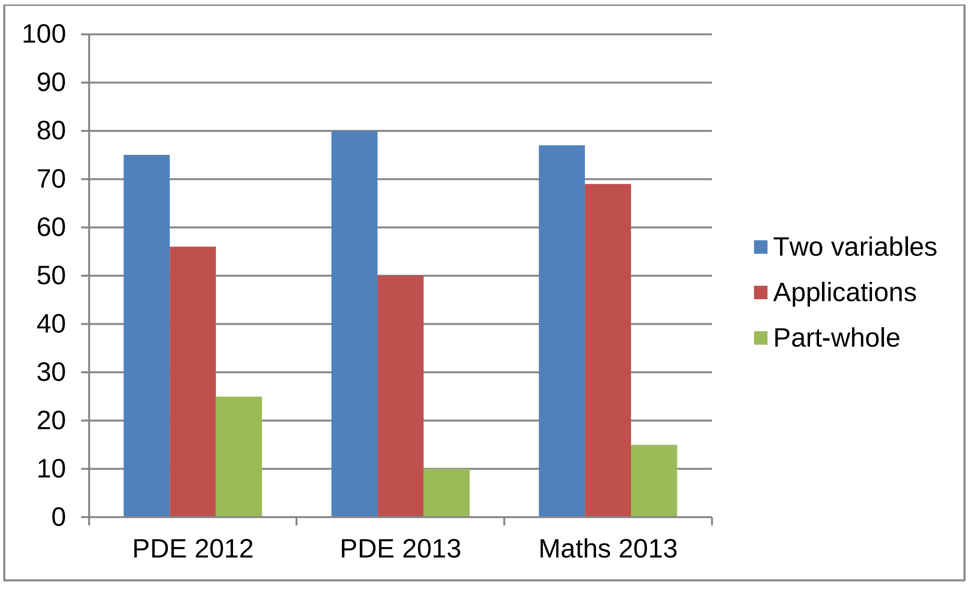
Year	Course	N
2012	PDE 2012	16
2013	PDE 2013	20
	Maths 2013	13

The meanings and representations offered by the three groups are discussed in turn.

Meanings of ratio

The meanings that participants explicitly ascribed to ratio occur in their responses to item 1. Figure 1 shows the distribution of the responses, classified by group, across the three themes (Table 1). Most students (three-quarters or more in each group) offered at least one meaning that reflected two variables, typically mentioning comparison or relationship or both. The PDE 2012 group had the highest percentage giving a part-whole meaning, but numbers are small in all cases (four PDE 2012 students, two PDE 2013 students and three Maths 2013 students). For all groups, the most usual response from the 'applications' theme was proportion, reflecting the Irish/English usage described above. [Figure 1 near here]

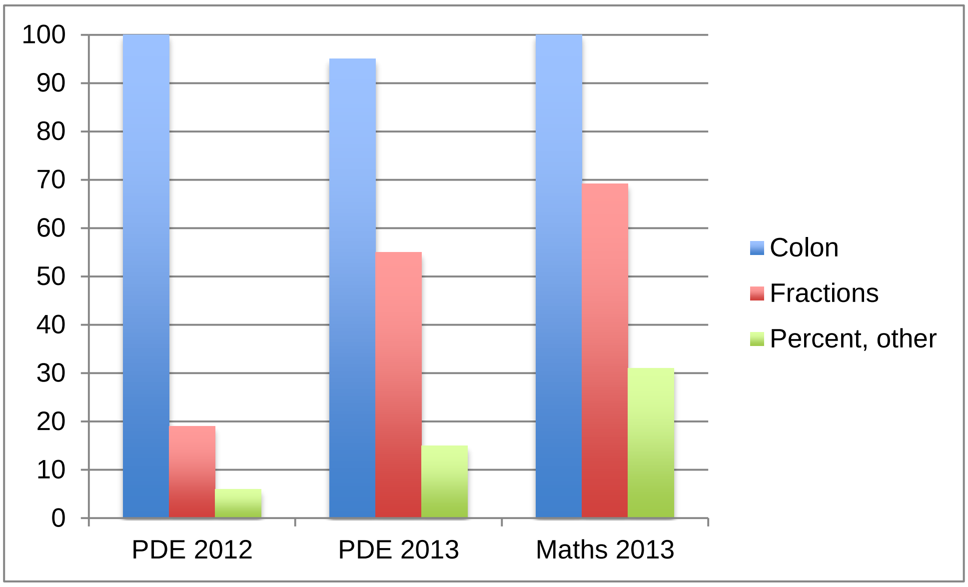
Fig. 1: Percentages of each group referring to each of the three emergent themes in responses to item 1



Representation by mathematical symbols

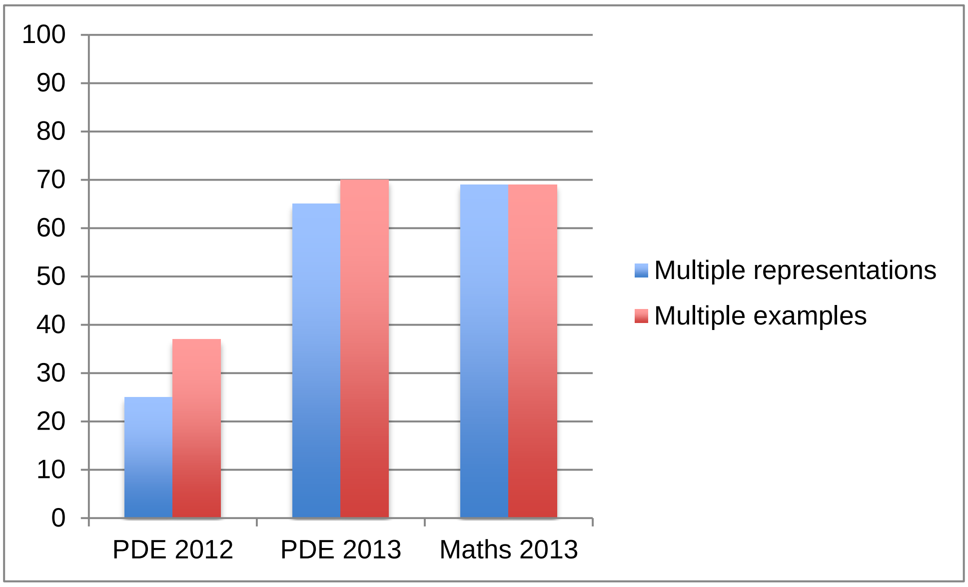
Again, the explicit responses, in this case to item 3, are counted, rather than symbolic representations used in responses to other items. Guided by the data, responses were classified as: colon; fractions (including use of the division slash and decimal notation); and percent and other responses. The distribution is shown in Figure 2. Overall, all but one of the participants used the colon notation, often as the main one; just three out of each of the 2013 groups gave priority to the 'fraction' notation. It should be noted that, in this case, there is no pejorative connotation in the 'fraction' classification; the notation (as opposed to a part-whole meaning) is fully acceptable. [Fig 2 near here]

Fig. 2: Percentages of each cohort using each main type of symbolic representation in responses to item 3



As noted earlier, this item taps knowledge of facts rather than understanding, but in its amended form it does ask participants to display knowledge of multiple representations. Figure 3 shows the percentages of each group offering multiple representations and multiple examples (the latter figure including multiple instances of the same representation). The two 2013 groups – responding to the altered item – produced more examples than the 2012 group. [Fig 3 near here]

Fig. 3: Percentages of each cohort using multiple representations / examples in responses to item 3



Representations for explanation and use

As shown in Figure 4, four of the PDE 2012 participants made no response to the original item 4. However, for the updated item, all 2013 participants responded, and most provided more than one example. [Fig 4 near here]

Fig. 4: Percentages of each cohort providing (multiple) examples in responses to item 4

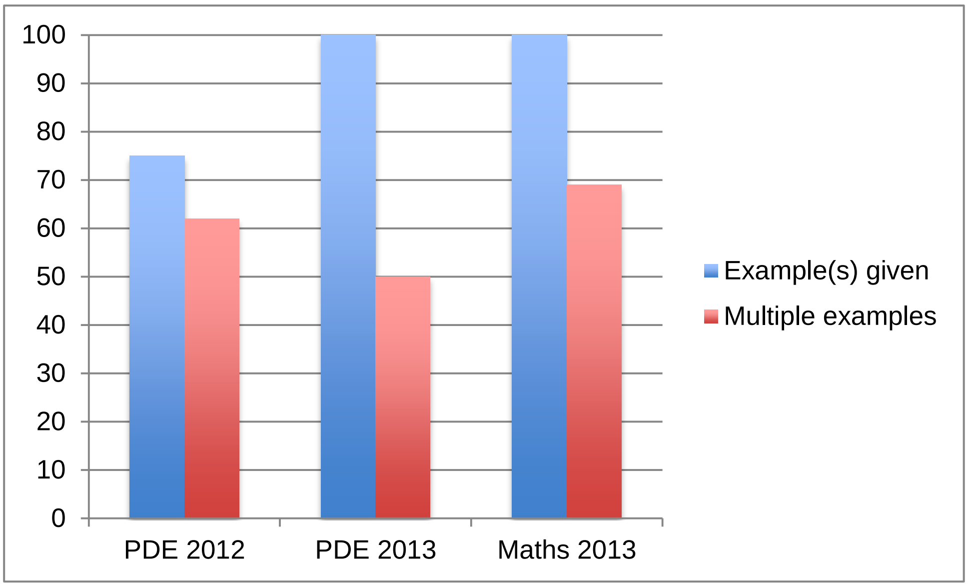


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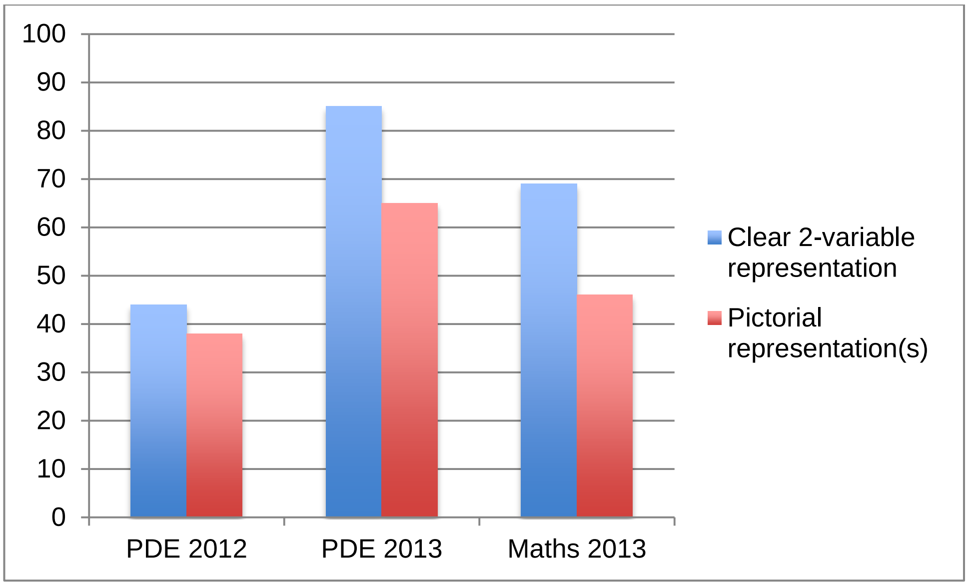
Figure 5 reports on two areas of interest: choice of a representation that clearly reflects a two-variable meaning for ratio, and provision of a pictorial representation (drawing, chart, and so forth). The figure shows the percentage of each group offering at least one example in each case. While the reformulated item explicitly allows verbal or symbolic representations, visual or graphic ones perhaps give more scope for imagination. Moreover, for some verbal explanations, insufficient detail was provided to show if a two-variable meaning was intended; this affects the percentage especially for the Maths 2013 group, who were the most inclined to use verbal explanations. [Fig 5 near here]

Fig. 5: Percentages of each cohort providing clear two-variable representations and pictorial representations in responses to item 4



Rather than providing further quantitative analysis of this small data set, it is appropriate here to include some of the responses to indicate their scope and style. They are presented in the Appendix. Figure A1 shows extreme examples from the PDE 2012 group: a rich pictorial response and one using only words or symbols. Figures A2 and A3 show examples from 2013 for which the participants provided respectively just one pictorial example and multiple examples, all being clear instances of the two-variable meaning of ratio. Figure A4 shows one of several extended responses from the Maths 2013 group. The predominance of middle-school examples in all figures reflects the overall pattern; the Maths 2013 group and PDE 2012 groups were more inclined than the PDE 2013 group to mention more advanced topics such as similar triangles (Fig. A2).

Answers to research questions

The refined research question A can now be answered as follows. The reformulated items elicited more responses: more symbolic representations for item 3, and more responses of various kinds for item 4. However, incorporation of the word 'explain' in item 4 may have skewed the responses towards verbal ones and suppressed some of the creativity shown in the 2012 drawings. The correct balance in the wording of this item still has to be struck. Subject to these exceptions, the styles of response of the two PDE groups were similar. The Maths 2013 group overall produced richer answers than the PDE groups, but this may chiefly reflect the greater time they devoted to the exercise rather than deeper knowledge. Perhaps a longer timeframe than the suggested 10 to 15 minutes would allow participants to tap into more of their knowledge and show their understanding.

This leads to discussion of research question B with regard to the ATEE Ratio Project conjecture on relational understanding. The authors did not find a way of using the responses as the basis of a metric measuring relational understanding reliably and validly; in particular, brief responses – using or omitting 'keywords' such as 'comparison' – could be poor indicators of depth or stability of understanding. One way forward, as suggested in the initial report (Berenson et al. 2013), would be to develop an interview protocol and explore selected participants' understanding in depth. Another would be to use the instrument as the basis for *class discussion*. Students might complete the instrument individually, but then share, clarify and deepen their understanding, as recommended by Pape and Tchoshanov (2001) – thus contributing to teacher education directly as well as via research.

Conclusion

It is clear that the topics of ratio and proportion require attention and research, particularly when mathematics teachers appear to be unsure in their own understanding of ratio and proportion. The ATEE Ratio Project, to which this paper contributes, aims to find the meanings and representations that prospective teachers associate with ratio, and hence to consider ways of describing their levels of understanding. In Ireland, ratio and proportion have been largely accepted as intuitively understood by students and teachers alike. However, analysis of the first round of data collected for the Ratio Project indicated that some of the small group of participant Irish pre-service teachers found difficulties in describing adequately what ratio means and in providing appropriate representations.

This paper describes a further round of data collection and analysis in Ireland, using an amended instrument that might encourage fuller responses. The two student groups – again small – participating in the exercise provided richer responses to the altered items. In many other respects the responses of the three groups were similar. A larger-scale Irish study is envisaged, both investigating the Irish situation further and opening up discussion on teaching and learning ratio to support recent curricular developments.

The paper also aspires to contribute to the ATEE Ratio Project by suggesting further refinements to the instrument and protocols for international use. Cultural differences in understanding of terminology will have to be taken into account. As well as interviews with participants, use of the instrument to promote discussion of ratio in teacher education classes may offer deeper insight into prospective teachers' understanding of this topic.

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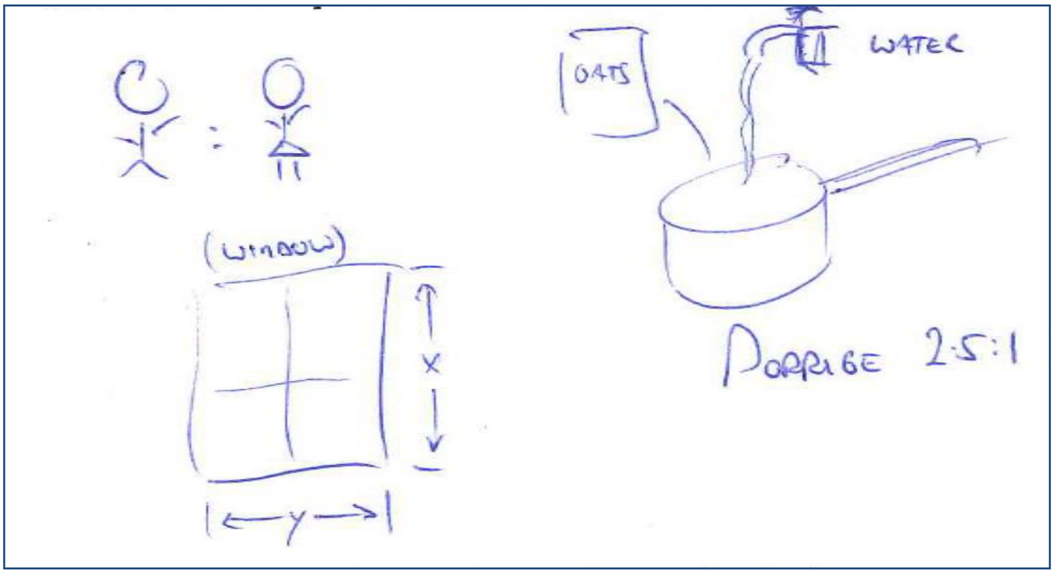
Appendix

[Fig A1 - A4 near here]

Fig. A1: responses to item 4 from the PDE 2012 group: 'Draw several representations of how ratios are used':



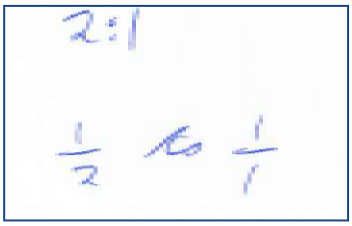
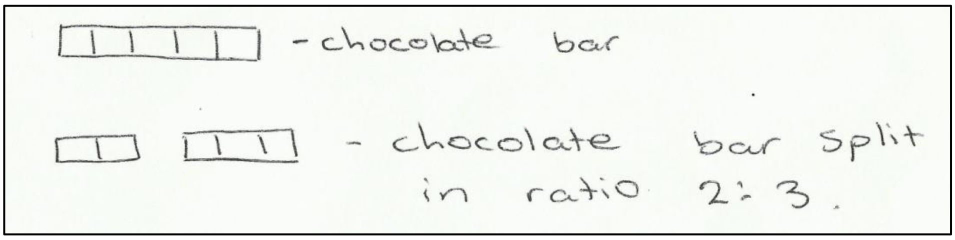
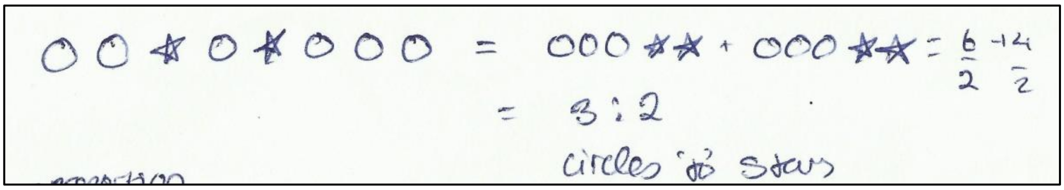


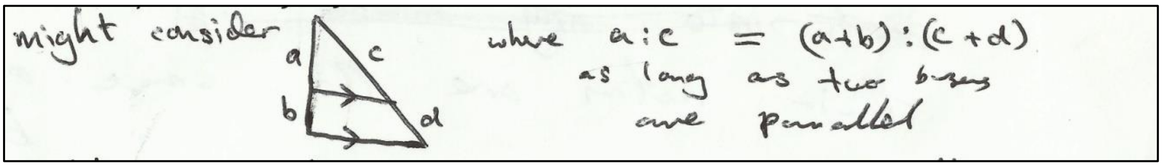
Fig. A2: Responses to item 4 from the 2013 groups: one pictorial representation offered (with explanations; other, non-pictorial, examples not shown)



(PDE 2013 – typical representation)

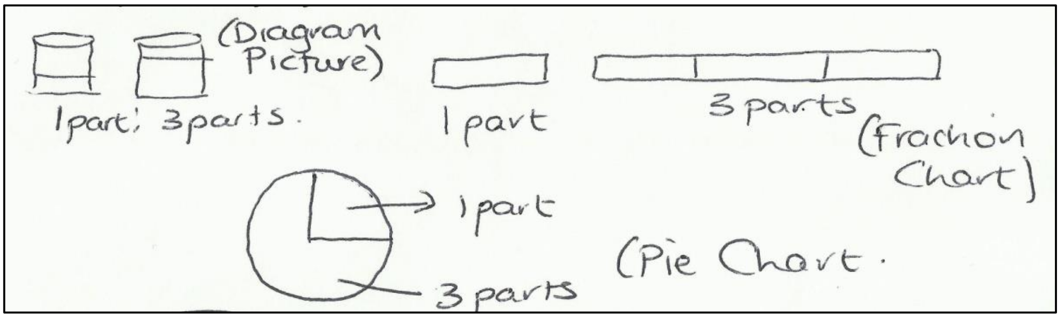


(PDE 2013 – unusual representation)



(Maths 2013)

Fig. A3: Responses to item 4 from the 2013 groups: clear two-variable representations – multiple representations and/or examples



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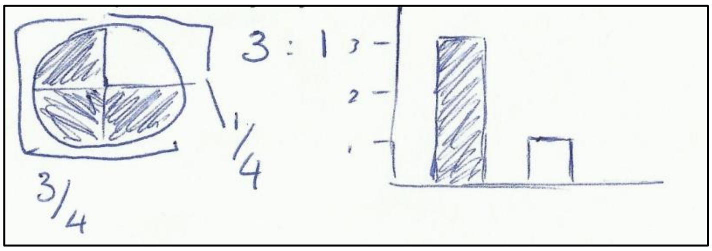
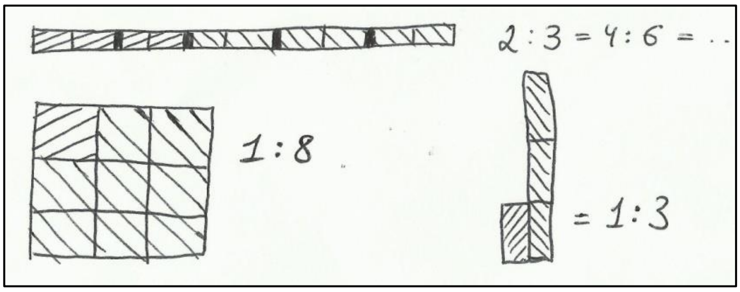
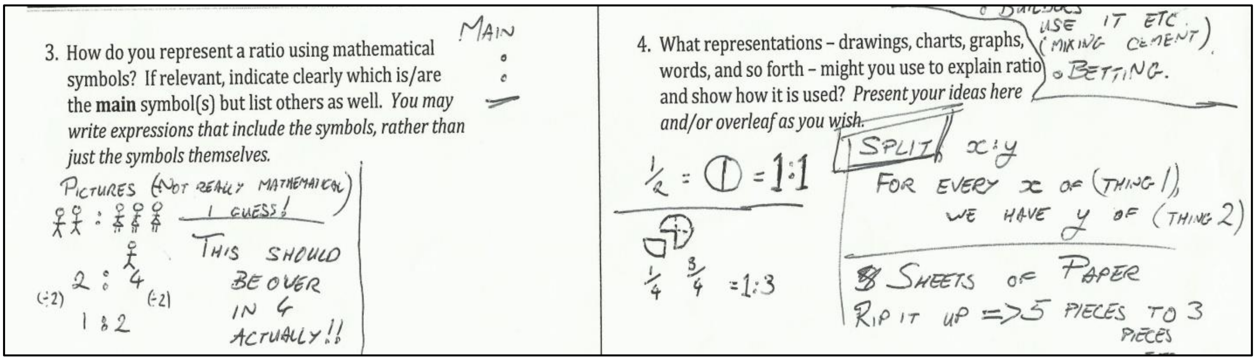


(PDE 2013)



(Maths 2013)

Fig. A4: response to item 4 from the Maths 2013 group



(Maths 2013)